Fractional Brownian motion in finance and queueing

Tommi Sottinen

Academic dissertation

To be presented, with the premission of the Faculty of Science of the University of Helsinki, for public criticism in Auditorium XIV of the Main Building of the University, on March 29th, 2003, at 10 o'clock a.m.

Department of Mathematics
Faculty of Science
University of Helsinki

Helsinki 2003

ISBN 952-91-5653-7 (Paperback)
ISBN 952-10-0984-5 (PDF)
Yliopistopaino
Helsinki 2003

Acknowledgements

I wish to express my sincere gratitude to my mentor Esko Valkeila for his guidance and for his help on my (unsuccesful) project to go to Jamaica.

Thanks are also due to Stefan Geiss and Paavo Salminen for reading the manuscript and for making useful suggestions.

My collaborators Yuriy Kozachenko, Esko Valkeila and Olga Vasylyk deserve my greatest gratitude. Without them I could not have written this thesis.

I would like to acknowledge interesting mathematical discussions with Ilkka Norros. I wish to thank Goran Peskir for his hospitality during my days at the University of Aarhus. In general, I would like to acknowledge the friendly atmosphere of the Stochastics group gathered around the Finnish Graduate School in Stochastics.

For financial support I am indebted with University of Helsinki, Academy of Finland, Finnish Graduate School in Stochastics and NorFA network "Stochastic analysis and its Applications".

I wish to thank my mother for her support.

Finally, I would like to thank my companion Saara, who has always been too loving and patient. Also, I would like to thank her for taking care of the evening cats when I have been taking care of this thesis.

On thesis

This thesis consists of two parts.

Part I is an introduction to the fractional Brownian motion and to the included articles. In Section 1 we consider briefly the (early) history of the fractional Brownian motion. In sections 2 and 3 we study some of its basic properties and provide some proofs. Regarding the proofs the author claims no originality. Indeed, they are mostly gathered from the existing literature. In sections 4 to 7 we recall some less elementary facts about the fractional Brownian motion that serve as background to the articles [a], [c] and [d]. The included articles are summarised in Section 8. Finally, Section 9 contains an errata of the articles.

Part II consists of the articles themselves:

[a] Sottinen, T. (2001) *Fractional Brownian motion, random walks and binary market models.* Finance Stoch. **5**, no. 3, 343–355.

[b] Kozachenko, Yu., Vasylyk, O. and Sottinen, T. (2002) *Path Space Large Deviations of a Large Buffer with Gaussian Input Traffic.* Queueing Systems **42**, no. 2, 113–129.

[c] Sottinen, T. (2002) *On Gaussian processes equivalent in law to fractional Brownian motion.* University of Helsinki, Department of Mathematics, Preprint **328**, 17 p. (submitted to Journal of Theoretical Probability)

[d] Sottinen, T. and Valkeila, E. (2002) *On arbitrage and replication in the Fractional Black–Scholes pricing model.* University of Helsinki, Department of Mathematics, Preprint **335**, 13 p. (submitted to Statistics and Decisions, under revision)

1. Story of process we nowadays call fractional Brownian motion

The fractional Brownian motion is a generalisation of the more well-known process of Brownian motion. It is a centred Gaussian process with stationary increments. However, the increments of the fractional Brownian motion are not independent, except in the standard Brownian case. The dependence structure of the increments is modeled by a parameter $H \in (0,1)$, viz. the covariance function $R = R_H$ of the fractional Brownian motion is

$$R(t,s) \;=\; \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right),$$

where $t, s \geq 0$.

The fractional Brownian motion was originally introduced by Kolmogorov [28], already in 1940. He was interested in modeling turbulence (see Kolmogorov [29], or Shiryaev [51] for more details of Kolmogorov's studies connected to turbulence). Kolmogorov did not use the name "fractional Brownian motion". He called the process "Wiener spiral". Kolmogorov studied the fractional Brownian motion within a Hilbert space framework and deduced its covariance function from a scaling property that we now call self-similarity.

On early works connected to fractional Brownian motion we would like to mention Hunt [25]. He was interested in almost sure convergence of random Fourier series and the modulus of continuity of such series. He also considered random Fourier transformations and their continuity properties. In his work the fractional Brownian motion was implicitly introduced as a Fourier–Wiener transformation of a power function (nowadays we would call this a spectral representation of the fractional Brownian motion). Hunt proved results concerning a Hölder-type modulus of continuity of the fractional Brownian motion.

Let us also note that Lévy [31] considered a process that is similar to the fractional Brownian motion. He introduced a process that is obtained from the standard Brownian motion as a fractional integral in the Riemann–Liouville sense. Although this process shares many of the (path) properties of the fractional Brownian motion it does not have stationary increments. This process is sometimes called the "Lévy fractional Brownian motion" or the "Riemann–Liouville process".

Yaglom [55] was interested in generalising the spectral theory of stationary processes to processes from a more general class. In particular, he was interested in linear extrapolation and linear filtering. Yaglom studied processes with "random stationary nth increments". In his work the fractional Brownian motion was considered as an example of a process with stationary first increments. It was defined through its spectral density.

Lamperti [30] studied semi-stable processes (which we nowadays call self-similar processes). The fractional Brownian motion appears implicitly in his work as an example of a Gaussian semi-stable process. Lamperti noted that the fractional Brownian motion cannot be Markovian, except in the standard

Brownian case. He showed that every self-similar process can be obtained from a stationary process and vice versa by a time-change transformation. Also, Lamperti proved a "fundamental limit theorem" stating that every non-degenerate self-similar process can be understood as a time-scale limit of a stochastic process.

Molchan and Golosov [36] studied the derivative of fractional Brownian motion using generalised stochastic processes (in the sense of Gel'fand–Ito). They called this derivative a "Gaussian stationary process with asymptotic power spectrum" (nowadays it is called fractional Gaussian noise or fractional white noise). Molchan and Golosov found a finite interval representation for the fractional Brownian motion with respect to the standard one (the more well-known Mandelbrot–Van Ness representation requires integration from minus infinity). In [36] there is also a reverse representation, i.e. a finite interval integral representation of the standard Brownian motion with respect to the fractional one. Molchan and Golosov noted the connection of these integral representations to deterministic fractional calculus. They also pointed out how one obtains the Girsanov theorem and prediction formulas for the fractional Brownian motion by using the integral representation.

The name "fractional Brownian motion" comes from the influential paper by Mandelbrot and Van Ness [34]. They defined the fractional Brownian motion as a fractional integral with respect to the standard one (whence the name). The notation for the index H and the current parametrisation with range $(0,1)$ are due to Mandelbrot and Van Ness also. The parameter H is called the Hurst index after an English hydrologist who studied the memory of Nile River maxima in connection of designing water reservoirs [26]. Mandelbrot and Van Ness considered an approximation of the fractional Gaussian noise by smoothing the fractional Brownian motion. They also studied simple interpolation and extrapolation of the smoothed fractional Gaussian noise and fractional Brownian motion.

Recently the fractional Brownian motion has found its way to many applications. It (and its further generalisations) has been studied in connection to financial time series, fluctuations in solids, hydrology, telecommunications and generation of artificial landscapes, just to mention few. Besides of these potential applications the study of the fractional Brownian motion is motivated from the fact that it is one of the simplest processes that is neighter a semimartingale nor a Markov process.

2. Fractional Brownian motion, self-similarity and long-range dependence

We define the fractional Brownian motion by its scaling property and discuss some basic properties of the process. A longer introduction to fractional Brownian motion can be found in the book by Samorodnitsky and Taqqu [49], Chapter 7.2 (which is surprising given the name of the book), or in a recent book by Embrechts and Maejima [21].

Definition 2.1. A process $X = (X_t)_{t\geq 0}$ is H-*self-similar* if

$$(X_{at})_{t\geq 0} \stackrel{d}{=} (a^H X_t)_{t\geq 0}$$

for all $a > 0$, where d means equality in distributions. The parameter $H > 0$ is called the *Hurst index*.

One might want to consider a seemingly more general notion of self-similarity, viz.

$$(X_{at})_{t\geq 0} \stackrel{d}{=} (bX_t)_{t\geq 0} \tag{2.1}$$

for some b depending on a. However, from (2.1) it follows that

$$b(a_1a_2)X_t \stackrel{d}{=} X_{a_1a_2t} \stackrel{d}{=} b(a_1)b(a_2)X_t.$$

Therefore, if X is non-trivial it follows that

$$b(a_1a_2) = b(a_1)b(a_2).$$

If, in addition, the process X is stochastically continuous at 0 it follows that $b(a) \leq 1$ for $a < 1$. Consequently, $b(a) = a^H$ for some $H \geq 0$. Furthermore, if $H = 0$ then from the stochastic continuity of X at 0 it follows that X is trivial. Indeed, for any $\varepsilon > 0$ and $a > 0$ we have

$$\begin{aligned}
\mathbf{P}(|X_t - X_0| > \varepsilon) &= \mathbf{P}(|X_{t/a} - X_0| > \varepsilon) \\
&= \lim_{a\to\infty} \mathbf{P}(|X_{t/a} - X_0| > \varepsilon) \\
&= 0.
\end{aligned}$$

Thus the power scaling with $H > 0$ in Definition 2.1 is indeed natural.

If X is a square integrable H-self-similar process it follows that

$$\mathbf{Var}\, X_t = \mathbf{Var}\, t^H X_1 = t^{2H}\mathbf{Var}\, X_1.$$

Assume further that X has stationary increments, zero mean and is normalised so that $\mathbf{Var}\, X_1 = 1$. Then we see that the covariance function R_H of X must be

$$R_H(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{2.2}$$

Of course, R_H might not be a proper covariance function for all $H > 0$, i.e. the process X might not exist. Indeed, suppose that $H > 1$. Then

$$\lim_{n\to\infty} \mathbf{Corr}(X_1, X_n - X_{n-1}) = \infty,$$

which is impossible.

If $H \in (0,1]$ then the corresponding process exists as the following lemma (taken from Samorodnitsky and Taqqu [49]) shows.

Lemma 2.2. *The function* R_H *is non-negative definite if* $H \in (0,1]$.

Proof. Let $t_1,\dots,t_n \ge 0$ and $u_1,\dots,u_n \in \mathbb{R}$. We want to show that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} R_H(t_i,t_j)u_iu_j \;\ge\; 0.$$

Set $t_0 := 0$ and add a value $u_0 := -\sum_{i=1}^n u_i$. Then $\sum_{i=0}^n u_i = 0$ and

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(t_i^{2H}+t_j^{2H}-|t_i-t_j|^{2H}\right)u_iu_j \;=\; -\sum_{i=0}^{n}\sum_{j=0}^{n}|t_i-t_j|^{2H}u_iu_j.$$

Since for any $\varepsilon > 0$ we have

$$\begin{aligned}\sum_{i=0}^{n}\sum_{j=0}^{n} e^{-\varepsilon|t_i-t_j|^{2H}}u_iu_j &= \sum_{i=0}^{n}\sum_{j=0}^{n}\left(e^{-\varepsilon|t_i-t_j|^{2H}}-1\right)u_iu_j \\ &= -\varepsilon\sum_{i=0}^{n}\sum_{j=0}|t_i-t_j|^{2H}u_iu_j \;+\; o(\varepsilon)\end{aligned}$$

as ε tends to zero it is sufficient to show that

$$\sum_{i=0}^{n}\sum_{j=0}^{n} e^{-\varepsilon|t_i-t_j|^{2H}}u_iu_j \;\ge\; 0.$$

But this follows from the fact that the mapping $\theta \mapsto e^{-\varepsilon|\theta|^{2H}}$ is a characteristic function for $H \in (0,1]$. □

Any non-negative definite function defines a unique zero mean Gaussian process. Thus, we can define the fractional Brownian motion to be the zero mean Gaussian process with covariance function R_H where $H \in (0,1]$. However, for $H = 1$

$$\begin{aligned}\mathbf{E}\,(X_t - tX_1)^2 &= \mathbf{E}\,X_t^2 - 2t\mathbf{E}\,X_tX_1 + t^2\mathbf{E}\,X_1^2 \\ &= (t^2 - 2t\cdot t + t^2)\mathbf{E}\,X_1^2 \\ &= 0.\end{aligned}$$

So $X_t = tX_1$ almost surely which is hardly interesting. Thus we shall exclude the case $H = 1$.

Definition 2.3. The *fractional Brownian motion* Z with Hurst index $H \in (0,1)$ is the unique zero mean H-self-similar Gaussian process with stationary increments and $\mathbf{E}\,Z_1^2 = 1$. Equivalently, it is the zero mean Gaussian process with covariance function (2.2).

Besides the self-similarity there is another property that makes the fractional Brownian motion a suitable model for many applications.

Definition 2.4. A stationary sequence $(X_n)_{n\in\mathbb{N}}$ of random variables is said to exhibit *long-range dependence* if the autocorrelation function $\rho(n) = \mathbf{E}\, X_0 X_n$ decays so slowly that

$$\sum_{n=1}^{\infty} \rho(n) \;=\; \infty.$$

If ρ decays exponentially, i.e. $\rho(n) \sim r^n$ as n tends to infinity, then the stationary sequence $(X_n)_{n\in\mathbb{N}}$ exhibits *short-range dependence*.

Actually there are many slightly different definitions for the long-range dependence. For details we refer to Beran [7].

Definition 2.5. The stationary sequence $(Y_n)_{n\in\mathbb{N}}$ where

$$Y_n \;:=\; Z_{n+1} - Z_n$$

and Z is a fractional Brownian motion with Hurst index H is called the *fractional Gaussian noise* with Hurst index H.

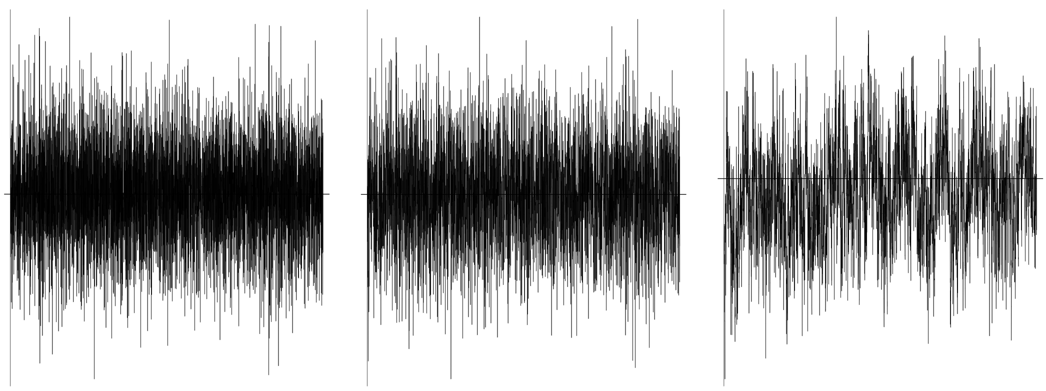

FIGURE 1. Simulated sample paths of a fractional Gaussian noise with Hurst indices $H = .1$ (left), $H = .5$ (middle) and $H = .9$ (right). The simulation was done by using the Conditionalised Random Midpoint Displacement method (and software) of Norros *et al.* [40].

The autocorrelation function $\rho = \rho_H$ of the fractional Gaussian noise with $H \neq \frac{1}{2}$ satisfies

$$\rho(n) \;\sim\; H(2H-1)\, n^{2H-2}$$

as n tends to infinity. Therefore, if $H > \frac{1}{2}$ then the increments of the corresponding fractional Brownian motion are positively correlated and exhibit the long-range dependence property. The case $H < \frac{1}{2}$ corresponds to negatively correlated increments and the short-range dependence. When $H = \frac{1}{2}$

the fractional Brownian motion is the standard Brownian motion, so it has independent increments.

Let us further illustrate the dependence structure of the fractional Brownian motion.

Proposition 2.6. *The fractional Brownian motion with Hurst index H is a Markov process if and only if $H = \frac{1}{2}$.*

Proof. It is well-known (cf. Kallenberg [27], Proposition 11.7) that a Gaussian process with covariance R is Markovian if and only if

$$R(s,u) \;=\; \frac{R(s,t)R(t,u)}{R(t,t)}$$

for all $s \leq t \leq u$. It is straightforward to check that the covariance function (2.2) satisfies the condition above if and only if $H = \frac{1}{2}$. $\square$

In what follows we shall consider the fractional Brownian motion on compact intervals, unless stated otherwise. Because of the self-similarity property we may and shall take that interval to be $[0,1]$.

3. Sample paths of fractional Brownian motion

Definition 3.1. A stochastic process $X = (X_t)_{t\in[0,1]}$ is *β-Hölder continuous* if there exists a finite random variable K such that

$$\sup_{s,t\in[0,1];s\neq t} \frac{|Z_t - Z_s|}{|t-s|^{\beta}} \;\leq\; K.$$

Proposition 3.2. *The fractional Brownian motion with Hurst index H admits a version with β-Hölder continuous sample paths if $\beta < H$. If $\beta \geq H$ then the fractional Brownian motion is almost surely not β-Hölder continuous on any time interval.*

Proof. The sufficiency of the condition $\beta < H$ is easy to prove. Indeed, let $n \in \mathbb{N}$. By self-similarity and stationarity of the increments we have

$$\mathbf{E}\left|Z_t - Z_s\right|^n \;=\; \mathbf{E}\left||t-s|^H Z_1\right|^n \;=\; |t-s|^{nH}\gamma_n,$$

where γ_n is the nth absolute moment of a standard normal random variable. The claim follows from this by the Kolmogorov–Chentsov criterion.

Consider the necessity of $\beta < H$. By stationarity of the increments it is enough to consider the point $t = 0$. By Arcones [4] the fractional Brownian motion satisfies the following law of the iterated logarithm:

$$\mathbf{P}\left(\limsup_{t\downarrow 0} \frac{Z_t}{t^H\sqrt{\ln\ln 1/t}} = 1\right) \;=\; 1.$$

Thus Z cannot be β-Hölder continuous for $\beta \geq H$ at any point $t \geq 0$. $\square$

In what follows we shall always use the Hölder continuous version of the fractional Brownian motion.

Corollary 3.3. *The fractional Brownian motion has almost surely nowhere differentiable sample paths.*

Proof. By stationarity of the increments it is enough to consider the time $t = 0$. If Z'_0 exists then

$$Z_s \;\; \leq \;\; (\varepsilon + Z'_0)s$$

for some positive $s \leq s_\varepsilon$. But this implies that Z is 1-Hölder continuous at 0. This contradicts the Proposition 3.2 above. □

Actually the non-differentiability is not connected to the Gaussian character of the fractional Brownian motion but follows from the self-similarity (cf. [34], Proposition 4.2).

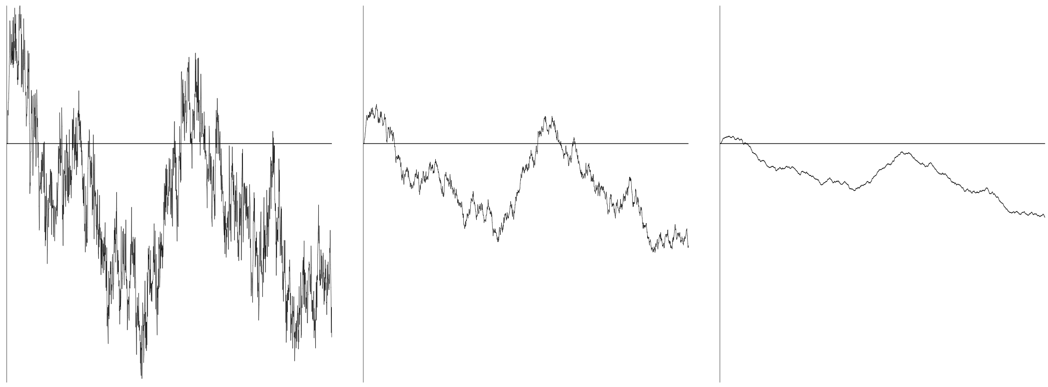

FIGURE 2. Sample paths of a fractional Brownian motion with Hurst indices $H = .25$ (left), $H = .5$ (middle) and $H = .75$ (right). The simulation was done using the Conditionalised Random Midpoint Displacement method (and software) of Norros *et al.* [40].

We shall introduce another notion of path regularity, the so-called p-variation. For details of p-variation and its connection to stochastic integration we refer to Dudley and Norvaiša [18, 19] and Mikosch and Norvaiša [35].

Consider partitions $\pi := \{t_k \,:\, 0 = t_0 < t_1 < \cdots < t_n = 1\}$ of $[0, 1]$. Denote by $|\pi|$ the mesh of π, i.e. $|\pi| := \max_{t_k \in \pi} \Delta t_k$ where $\Delta t_k := t_k - t_{k-1}$. Let f be a function over the interval $[0, 1]$. Then for $p \in [1, \infty)$

$$v_p(f;\pi) \;\; := \;\; \sum_{t_k \in \pi} |\Delta f(t_k)|^p$$

where $\Delta f(t_k) := f(t_k) - f(t_{k-1})$ is the p*-variation of* f *along the partition* π.

Definition 3.4. Let f be a function over the interval $[0,1]$. If

$$v_p^0(f) \ := \ \lim_{|\pi|\to 0} v_p(f;\pi)$$

exists we say that f has *finite p-variation.* If

$$v_p(f) \ := \ \sup_{\pi} v_p(f;\pi)$$

is finite then f has *bounded p-variation.* The *variation index* of f is

$$v(f) \ := \ \inf\{p>0 \,:\, v_p(f)<\infty\}$$

where the infimum of an empty set is ∞.

It is obvious from the definition that $v_p(f) \geq v_p^0(f)$.

Definition 3.5. The Banach space $\mathscr{W}_p$ is the set of functions of bounded p-variation equipped with the norm

$$\|f\|_{[p]} \ := \ \|f\|_{(p)} + \|f\|_{\infty},$$

where $\|f\|_{(p)} := v_p(f)^{1/p}$ and $\|f\|_\infty := \sup_{t\in[0,1]}|f(t)|$.

When $p=2$ the finite 2-variation v_2^0 coincides with the classical notion of quadratic variation in the martingale theory. When $p=1$ the bounded 1-variation v_1 is the usual bounded variation.

Hölder continuity is closely related to the bounded p-variation.

Lemma 3.6. *Let $p\in[1,\infty)$ and let f be a function over the interval $[0,1]$. Then f has bounded p-variation if and only if*

$$f \ = \ g\circ h$$

where h is a bounded non-negative increasing function on $[0,1]$ and g is $1/p$-Hölder continuous function defined on $[h(0),h(1)]$.

Proof. Consider the if part. Take h to be the identity function and suppose that f is $1/p$-Hölder continuous with Hölder constant K. Then for any partition π of the interval $[0,1]$ we have

$$\sum_{t_k\in\pi}|\Delta f(t_k)|^p \ \leq \ K^p\sum_{t_k\in\pi}|\Delta t_k|^{\frac{1}{p}\cdot p} \ = \ K^p.$$

So $f\in\mathscr{W}_p$.

For the only if part suppose that $f\in\mathscr{W}_p$. Let $h(x)$ be the p-variation of f on $[0,x]$. Then h is a bounded increasing function. Moreover

$$|f(x)-f(y)|^p \ \leq \ |h(x)-h(y)|$$

since p-variation is subadditive with respect to intervals. Now define g on $\{h(x) : x \in [0,1]\}$ by $g(h(x)) := f(x)$ and extend it to $[h(0), h(1)]$ by linearity. Obviously g is $1/p$-Hölder continuous. □

For the fractional Brownian motion with Hurst index H the critical value for p-variation is $1/H$ as the following lemma suggests.

Lemma 3.7. *Set $\pi_n := \{t_k = \frac{k}{n} : k = 1, \dots, n\}$ and let Z be a fractional Brownian motion with Hurst index H. Denote by γ_p the p th absolute moment of a standard normal random variable. Then*

$$\lim_{n\to\infty} v_p(Z;\pi_n) = \begin{cases} \infty, & \text{if } p < 1/H \\ \gamma_p, & \text{if } p = 1/H \\ 0, & \text{if } p > 1/H \end{cases}$$

where the limit is understood in the mean square sense.

Proof. By the self-similarity property we have

$$\begin{aligned} \sum_{t_k\in\pi_n} |\Delta Z_{t_k}|^p &\stackrel{d}{=} \sum_{t_k\in\pi_n} |\Delta t_k|^{pH} \, |Z_k - Z_{k-1}|^p \\ &= n^{pH-1}\frac{1}{n}\sum_{k=1}^{n} |Z_k - Z_{k-1}|^p \,. \end{aligned}$$

Now by Proposition 7.2.9 of [49] the stationary sequence $(Z_k - Z_{k-1})_{k\in\mathbb{N}}$ has spectral density with respect to the Lebesgue measure. Therefore by Theorem 14.2.1 of [10] it is ergodic. The claim follows from this. □

In Lemma 3.7 the choice of the special sequence $(\pi_n)_{n\in\mathbb{N}}$ of equidistant partitions was crucial.

Proposition 3.8. *Let Z be a fractional Brownian motion with Hurst index H. Then $v_p^0(Z) = 0$ almost surely if $p > 1/H$. For $p < 1/H$ we have $v_p(Z) = \infty$ and $v_p^0(Z)$ does not exist. Moreover $v(Z) = 1/H$.*

Proof. Denote by K the Hölder constant of the fractional Brownian motion. Let $p > 1/H$ and let π be a partition of $[0,1]$. Then by Proposition 3.2

$$\begin{aligned} \sum_{t_k\in\pi} |\Delta Z_{t_k}|^p &\le \sum_{t_k\in\pi} \left|K|\Delta t_k|^{\beta}\right|^p \\ &= K^p \sum_{t_k\in\pi} |\Delta t_k|^{\beta p} \\ &\le K^p|\pi| \sum_{t_k\in\pi} |\Delta t_k|^{\beta p-1} \end{aligned}$$

almost surely for any $\beta < H$. Letting $|\pi|$ tend to zero we see that $v_p^0(Z) = 0$ almost surely.

Suppose then that $p < 1/H$. Then by Lemma 3.7 we can choose a subsequence $(\pi'_n)_{n\in\mathbb{N}}$ of the sequence of equidistant partitions $(\pi_n)_{n\in\mathbb{N}}$ such that $v_{1/H}(Z;\pi'_n)$ converges almost surely to $\gamma_{1/H}$. Consequently, along this subsequence we have $\lim_{n\to\infty} v_p(Z;\pi'_n) = \infty$ almost surely. Since $|\pi'_n|$ tends to zero as n increases $v^0_p(Z)$ cannot exist. This also shows that $v_p(Z) = \infty$ almost surely for $p < 1/H$.

Finally since $v_p(Z)$ is finite almost surely for all $p > 1/H$ by Lemma 3.6 and Proposition 3.2, we must have $v(Z) = 1/H$. □

Finally we are ready to prove the fact that makes stochastic integration with respect to the fractional Brownian motion an interesting problem.

Corollary 3.9. *The fractional Brownian motion with Hurst index $H \neq \frac{1}{2}$ is not a semimartingale.*

Proof. For $H < \frac{1}{2}$ we know by Proposition 3.8 that the fractional Brownian motion has no quadratic variation. So it cannot be a semimartingale.

Suppose then that $H > \frac{1}{2}$ and assume that the fractional Brownian motion is a semimartingale with decomposition $Z = M + A$. Now Proposition 3.8 states that Z has zero quadratic variation. So the martingale $M = Z - A$ has zero quadratic variation. Since Z is continuous we know by the properties of the semimartingale decomposition that M is also continuous. But a continuous martingale with zero quadratic variation is a constant. So $Z = A + M_0$ and Z must have bounded variation. This is a contradiction since $v_1(Z) \geq v_p(Z) = \infty$ for all $p < 1/H$. □

4. Fractional calculus and integral representations of fractional Brownian motion

The fractional Brownian motion may be considered as a fractional integral of the white noise (the formal derivative of the standard Brownian motion). So we take a short detour to deterministic fractional calculus. A comprehensive treatment of the subject can be found in the book by Samko *et al.* [48]. For discussion on the connection between the integral representations and the fractional calculus we refer to Pipiras and Taqqu [44, 45].

The starting point of fractional calculus is the well-known formula for the iterated integral

$$\int_a^{t_n} \cdots \int_a^{t_2} f(t_1)\,\mathrm{d}t_1 \cdots \mathrm{d}t_{n-1} \quad = \quad \frac{1}{(n-1)!} \int_a^{t_n} \frac{f(s)}{(t_n - s)^{1-n}}\,\mathrm{d}s. \tag{4.1}$$

Since $(n-1)! = \Gamma(n)$ the right hand side of (4.1) makes sense for non-integer n. Denote $x_\pm := \max(\pm x, 0)$.

Definition 4.1. Let f be a function over $[0,1]$ and $\alpha > 0$. The integrals

$$\mathrm{I}^{\alpha}_{\pm} f\,(t) \;\;:=\;\; \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{f(s)}{(s-t)_{\mp}^{1-\alpha}}\,\mathrm{d}s$$

are called the left-sided and right-sided Riemann–Liouville *fractional integrals* of order α; $\mathrm{I}^0_{\pm}$ are identity operators.

The fractional derivatives are defined by the formal solutions of the Abel integral equations

$$\mathrm{I}^{\alpha}_{\pm} g \;\;=\;\; f.$$

For the derivation see [48], Section 2.1.

Definition 4.2. Let f be a function over $[0,1]$ and $\alpha \in (0,1)$. Then

$$\mathrm{D}^{\alpha}_{\pm} f\,(t) \;\;:=\;\; \frac{\pm 1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \frac{f(s)}{(s-t)_{\mp}^{\alpha}}\,\mathrm{d}s.$$

are the left-sided and right-sided Riemann–Liouville *fractional derivatives* of order α; $\mathrm{D}^1_{\pm}$ are ordinary derivative operators and $\mathrm{D}^0_{\pm}$ are identity operators.

If one ignores the difficulties related to divergent integrals and formally changes the order of differentiation and integration in the definition of the fractional derivatives one obtains that

$$\mathrm{I}^{\alpha}_{\pm} \;\;=\;\; \mathrm{D}^{-\alpha}_{\pm}.$$

We shall take the above as a definition for fractional integral of negative order and use the obvious unified notation.

Fractional integrals satisfy a semigroup property (cf. [48], Theorem 2.5).

Lemma 4.3. *The composition formula*

$$\mathrm{I}^{\alpha}_{\pm}\mathrm{I}^{\beta}_{\pm} f \;\;=\;\; \mathrm{I}^{\alpha+\beta}_{\pm} f$$

is valid in any of the following cases:

(i) $\beta \geq 0,\;\; \alpha+\beta \geq 0$ *and* $f \in L^1([0,1])$,
(ii) $\beta \leq 0,\;\; \alpha \geq 0$ *and* $f \in \mathrm{I}^{-\beta}_{\pm} L^1([0,1])$,
(iii) $\alpha \leq 0,\;\; \alpha+\beta \leq 0$ *and* $f \in \mathrm{I}^{-\alpha-\beta}_{\pm} L^1([0,1])$.

We have a *fractional integration by parts* formula (cf. [48], p. 34 and p. 46).

Lemma 4.4. *Suppose that* $\alpha > 0$. *Then*

$$\int_0^1 f(t)\,\mathrm{I}^{\alpha}_{+} g\,(t)\,\mathrm{d}t \;\;=\;\; \int_0^1 \mathrm{I}^{\alpha}_{-} f\,(t)\,g(t)\,\mathrm{d}t \tag{4.2}$$

is valid if $f \in L^p([0,1])$ *and* $g \in L^q([0,1])$ *where* $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ *and* $p \geq 1$, $q \geq 1$ *with* $p \neq 1$, $q \neq 1$ *if* $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$. *If* $\alpha \in (-1, 0)$ *then (4.2) holds for* $f \in \mathrm{I}_-^{-\alpha} L^p([0,1])$ *and* $g \in \mathrm{I}_+^{-\alpha} L^q([0,1])$ *with* $\frac{1}{p} + \frac{1}{q} \leq 1 - \alpha$.

Let us introduce fractional integrals and derivatives over the real line.

Definition 4.5. Let f be a function over $\mathbb{R}$ and $\alpha > 0$. Then the integrals

$$\mathbf{I}_\pm^\alpha f\,(t) \;:=\; \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} \frac{f(s)}{(t-s)_\pm^{1-\alpha}} \,\mathrm{d}s$$

are called the left-sided and right-sided *fractional integrals* of order α; $\mathbf{I}_\pm^0$ are identity operators.

Definition 4.6. Let f be a function over $\mathbb{R}$ and $\alpha \in (0,1)$. Then

$$\mathbf{D}_\pm^\alpha f\,(t) \;:=\; \lim_{\varepsilon \to 0} \frac{\alpha}{1-\alpha} \int_{\varepsilon}^{\infty} \frac{f(t) - f(t \mp s)}{s^{1+\alpha}} \,\mathrm{d}s$$

are the left-sided and right-sided *Marchaud fractional derivatives* of order α; $\mathbf{D}_\pm^0$ are identity operators and $\mathbf{D}_\pm^1$ are ordinary derivative operators.

For $\alpha \in (-1,0)$ we shall denote

$$\mathbf{I}_\pm^\alpha \;:=\; \mathbf{D}_\pm^{-\alpha}.$$

A fractional integration by parts formula

$$\int_{-\infty}^{\infty} f(t)\mathbf{I}_+^\alpha g\,(t)\,\mathrm{d}t \;=\; \int_{-\infty}^{\infty} \mathbf{I}_-^\alpha f\,(t)\, g(t)\,\mathrm{d}t \tag{4.3}$$

is valid for "sufficiently good" functions f and g (cf. [48] p. 96).

Let us consider now integral representations of the fractional Brownian motion. Define kernels $z = z_H$ and $z^* = z_H^*$ on $[0,1]^2$ as

$$z(t,s) \;:=$$
$$c_H \left(\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H - \tfrac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} \,\mathrm{d}u \right),$$

$$z^*(t,s) \;:=$$
$$c_H' \left(\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{\frac{1}{2}-H} - (H - \tfrac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{\frac{1}{2}-H} \,\mathrm{d}u \right).$$

Here c_H and c'_H are the normalising constants

$$c_H := \sqrt{\frac{(2H+\frac{1}{2})\Gamma(\frac{1}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}},$$

$$c'_H := \frac{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}{\mathrm{B}(\frac{1}{2}-H, H+\frac{1}{2})\sqrt{(2H+\frac{1}{2})\Gamma(\frac{1}{2}-H)}}$$

where Γ and B denote the gamma and beta functions, respectively. The kernels z and z^* are of Volterra type, i.e. they vanish whenever the second argument is greater than the first one.

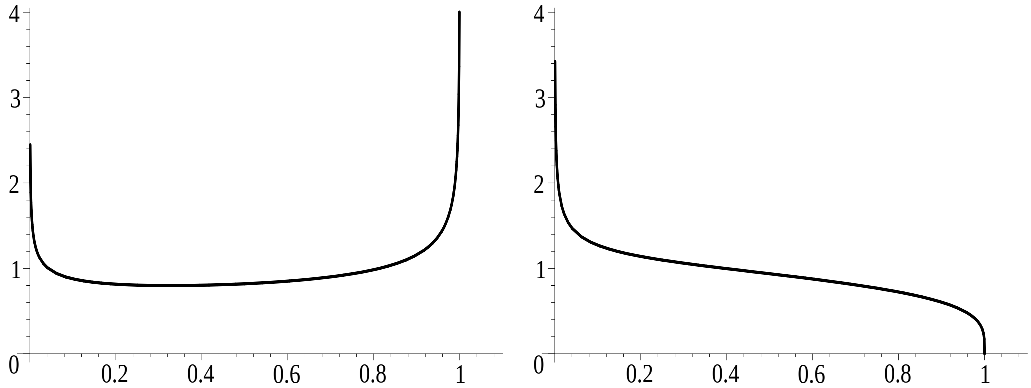

FIGURE 3. Kernel $z(1, s)$ with Hurst indices $H = .25$ (left) and $H = .75$ (right).

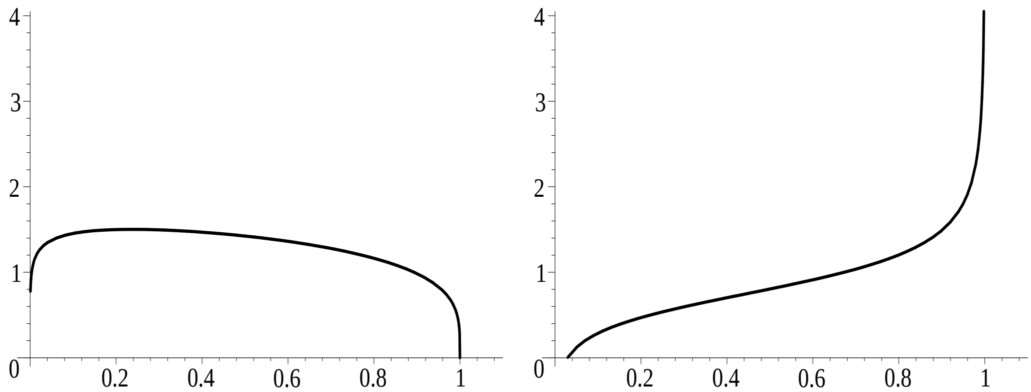

FIGURE 4. Approximative graphs of the "resolvent" kernel $z^*(1, s)$ with Hurst indices $H = .25$ (left) and $H = .75$ (right)

Theorem 4.7. *Let W be a standard Brownian motion. Then the process Z defined as the Wiener integral of the kernel $z = z_H$*

$$Z_t \ := \ \int_0^t z(t,s)\,\mathrm{d}W_s \tag{4.4}$$

is a fractional Brownian motion with Hurst index H. If Z is a fractional Brownian motion with Hurst index H then a Brownian motion W can be constructed as the fractional Wiener integral of the kernel $z^* = z^*_H$

$$W_t \ := \ \int_0^t z^*(t,s)\,\mathrm{d}Z_s. \tag{4.5}$$

The integrals (4.4) and (4.5) can be understood in the $L^2(\Omega)$-sense as well as in the pathwise sense as improper Riemann–Stieltjes integrals.

Representations (4.4) and (4.5) are due to Molchan and Golosov [36]. Later they have appeared in different forms in e.g. Decreusefond and Üstünel [15], Norros *et al.* [41] and Nuzman and Poor [43].

The connection to fractional calculus and the representations (4.4) and (4.5) is the following. Consider the weighted fractional integral operators

$$\begin{aligned}
\mathrm{K}f(t) \ &:= \ C_H t^{\frac{1}{2}-H}\left(\mathrm{I}_-^{H-\frac{1}{2}} s^{H-\frac{1}{2}} f(s)\right)(t), \\
\mathrm{K}^*f(t) \ &:= \ \frac{1}{C_H} t^{\frac{1}{2}-H}\left(\mathrm{I}_-^{\frac{1}{2}-H} s^{H-\frac{1}{2}} f(s)\right)(t)
\end{aligned}$$

where

$$C_H \ := \ \sqrt{\frac{2H(H-\frac{1}{2})\Gamma(H-\frac{1}{2})^2}{\mathrm{B}(H-\frac{1}{2}, 2-2H)}}.$$

Then we have

$$\begin{aligned}
z(t,s) \ &= \ \mathrm{K}\mathbf{1}_{[0,t]}(s), \\
z^*(t,s) \ &= \ \mathrm{K}^*\mathbf{1}_{[0,t]}(s).
\end{aligned}$$

Mandelbrot and Van Ness [34] constructed the fractional Brownian motion on the whole real line.

Theorem 4.8. *Let W be the standard Brownian motion on the real line. Then the process $(Z_t)_{t\in\mathbb{R}}$ defined as*

$$Z_t \ := \ \int_{-\infty}^{\infty} f(t,s)\,\mathrm{d}W_s \tag{4.6}$$

where

$$f(t,s) \ := \ c_H\left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right),$$

is a fractional Brownian motion with Hurst index H.

Representation (4.6) may also be inverted. See Pipiras and Taqqu [44] for details.

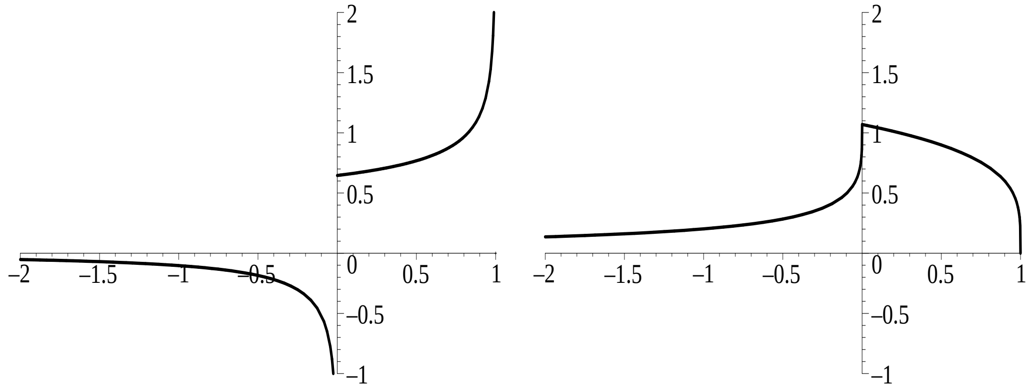

FIGURE 5. Kernel $f(1,s)$ with Hurst indices $H = .25$ (left) and $H = .75$ (right).

Let us end this section with heuristically justifying the name *fractional* Brownian motion. The kernel f can be written as the fractional integral

$$f(t,s) \;=\; \Gamma(H+\tfrac{1}{2})c_H \left(\mathbf{I}_-^{H-\frac{1}{2}}\mathbf{1}_{[0,t]}\right)(s).$$

Denote by $\dot{Z}$ and $\dot{W}$ the (non-existent) derivatives of the fractional Brownian motion and the standard one, respectively. Let us omit the constant $\Gamma(H+\frac{1}{2})c_H$. The fractional integration by parts formula (4.3) yields

$$\begin{aligned} Z_t &= \int_{-\infty}^{\infty} \left(\mathbf{I}_-^{H-\frac{1}{2}}\mathbf{1}_{[0,t]}\right)(s)\dot{W}_s \,\mathrm{d}s \\ &= \int_{-\infty}^{\infty} \mathbf{1}_{[0,t]}(s)\left(\mathbf{I}_+^{H-\frac{1}{2}}\dot{W}\right)_s \mathrm{d}s \\ &= \int_0^t \left(\mathbf{I}_+^{H-\frac{1}{2}}\dot{W}\right)_s \mathrm{d}s \\ &= \left(\mathbf{I}_+^{H+\frac{1}{2}}\dot{W}\right)_t . \end{aligned}$$

This means that a fractional Brownian motion with index H is obtained by integrating the white noise (on the real line) $H+\frac{1}{2}$ times. Similar heuristics with the representation (4.4) yields

$$\dot{Z}_t \;=\; t^{\frac{1}{2}-H}\left(\mathrm{I}_+^{H-\frac{1}{2}} s^{H-\frac{1}{2}}\dot{W}_s\right)_t .$$

Here the white noise $\dot{W}$ is given on the interval $[0,1]$.

5. Wiener integrals

We consider integrals with respect to the fractional Brownian motion where the integrand is a deterministic function. It turns out that even this most simple form of stochastic integration involves difficulties. Indeed, Pipiras and Taqqu [45] showed that for $H > \frac{1}{2}$ the space of integrands that appears naturally (see Definition 5.1 below) is not complete (in the case $H \leq \frac{1}{2}$ it is complete).

Suppose that $f \in \mathscr{E}$, i.e.

$$f \;=\; \sum_{k=1}^{n} a_k \mathbf{1}_{(t_{k-1},t_k]}$$

where $0 = t_0 < t_1 < \cdots < t_n = 1$ and $a_1, \ldots, a_n \in \mathbb{R}$. In this case it is natural to set

$$\int_0^1 f(t)\,\mathrm{d}Z_t \;:=\; \sum_{k=1}^{n} a_k \Delta Z_{t_k}.$$

Now integral representation (4.4) yields

$$\int_0^1 f(t)\,\mathrm{d}Z_t \;=\; \int_0^1 \mathrm{K}f(t)\,\mathrm{d}W_t \tag{5.1}$$

for any $f \in \mathscr{E}$. Similarly, for $f \in \mathscr{E}$, (4.5) yields

$$\int_0^1 f(t)\,\mathrm{d}W_t \;=\; \int_0^1 \mathrm{K}^* f(t)\,\mathrm{d}Z_t \tag{5.2}$$

The equations (4.4) and (4.5) can be understood "ω-by-ω".

Since the classical Wiener integral is defined for any $f \in L^2([0,1])$, the equality (5.1) leads to the following definition.

Definition 5.1. Set

$$\Lambda \;:=\; \left\{ f \,:\, \mathrm{K}f \in L^2([0,1]) \right\}.$$

Then for $f \in \Lambda$ the *Wiener integral* of f with respect to fractional Brownian motion Z is

$$\int_0^1 f(t)\,\mathrm{d}Z_t \;:=\; \int_0^1 \mathrm{K}f\,(t)\,\mathrm{d}W_t.$$

The integral of Definition 5.1 can be considered as a limit of elementary functions. Indeed, theorems 4.1 and 4.2 of Pipiras and Taqqu [45] state the following.

Theorem 5.2. *For any $H \in (0,1)$ the class of functions Λ is a linear space with inner product*

$$\langle f, g\rangle_\Lambda \;:=\; \langle \mathrm{K}f, \mathrm{K}g\rangle_{L^2([0,1])}\,.$$

Moreover, $\mathscr{E}$ is dense in Λ.

Relations (4.4) and (4.5) together with the semigroup property of the fractional integrals (Lemma 4.3) imply the following.

Lemma 5.3. *The equalities*

$$\mathrm{KK}^* f \;=\; f \;=\; \mathrm{K}^*\mathrm{K} f$$

hold for any $f \in \mathscr{E}$. *If* $H > \frac{1}{2}$ *then the equality*

$$\mathrm{K}^*\mathrm{K} f \;=\; f$$

holds for any $f \in L^2([0,1])$. *If* $H < \frac{1}{2}$ *then, for any* $f \in L^2([0,1])$,

$$\mathrm{KK}^* f \;=\; f.$$

Lemma 5.3 cannot be extended. Indeed, Pipiras and Taqqu ([45], Lemma 5.3) showed the following.

Lemma 5.4. *Let* $H > \frac{1}{2}$. *Then there exist functions* $f \in L^2([0,1])$ *such that the equation*

$$\mathrm{K} g \;=\; f$$

has no solution in g.

The idea behind Lemma 5.4 is that for $H > \frac{1}{2}$, K is a fractional *integral* operator. Consequently, the function $\mathrm{K}g$ must be "smooth". However let

$$f(t) \;:=\; t^{-\alpha}\psi(t),$$

where ψ is the real part of the Weierstrass function

$$\psi^*(t) \;=\; \sum_{n=1}^{\infty} 2^{-\frac{\alpha}{2}n} e^{i2^n t}.$$

One can show that f does not belong to the image of K.

Let us recall the concepts of the linear space and reproducing kernel Hilbert space of a stochastic process.

Definition 5.5. The *linear space* $\mathscr{H}_1$ of a process Z is the closure in $L^2(\Omega)$ of the random variables F of the form

$$F \;=\; \sum_{k=1}^{n} a_k Z_{t_k},$$

where $n \in \mathbb{N}$, $a_k \in \mathbb{R}$ and $t_k \in [0,1]$ for $k = 1,\ldots,n$.

Definition 5.6. The *reproducing kernel Hilbert space* $\mathscr{R}$ of Z with covariance function R is the closure of $\operatorname{span}\{R(t,\cdot) : t \in [0,1]\}$ with respect to the inner product

$$\langle R(t,\cdot), R(s,\cdot)\rangle_{\mathscr{R}} \;:=\; R(t,s).$$

So $\mathscr{H}_1$ is the set of random variables that can be approximated in $L^2(\Omega)$ by Wiener integrals. Naturally one wants to identify any $F \in \mathscr{H}_1$ with a single function $f \in \Lambda$ so that

$$F = \int_0^1 f(t)\,\mathrm{d}Z_t.$$

This is possible if and only if Λ is complete. Otherwise Λ is isometric to a proper subspace of $\mathscr{H}_1$. The space $\mathscr{R}$ is complete and the mapping $R(t,\cdot) \mapsto Z_t$ extends to an isometry between $\mathscr{R}$ and $\mathscr{H}_1$.

By Lemma 5.3 and Lemma 5.4 we have the following.

Proposition 5.7. *If $H \leq \frac{1}{2}$ then*

$$\Lambda = \left\{ \mathrm{K}^* f \,:\, f \in L^2([0,1]) \right\}.$$

Moreover, the inner product space Λ is complete and hence isometric to $\mathscr{H}_1$. If $H > \frac{1}{2}$ then the inner product space Λ is not complete and hence isometric to a proper subspace of $\mathscr{H}_1$.

The space $\mathscr{R}$ can be described in the following way (cf. Decreusefond and Üstünel [15], Theorem 3.3 and Remark 3.1).

Proposition 5.8. *A function $f \in \mathscr{R}$ if and only if it can be represented as*

$$f(t) = \int_0^t z(t,s)\tilde{f}(s)\,\mathrm{d}s$$

for some $\tilde{f} \in L^2([0,1])$. The scalar product $\langle \cdot,\cdot \rangle_{\mathscr{R}}$ on $\mathscr{R}$ is given by

$$\langle f, g \rangle_{\mathscr{R}} = \langle \tilde{f}, \tilde{g} \rangle_{L^2([0,1])}.$$

Moreover, as a vector space

$$\mathscr{R} = \mathrm{I}_+^{H+\frac{1}{2}} L^2([0,1]).$$

6. Pathwise integrals

Although the fractional Brownian motion has almost surely sample paths of unbounded variation one can define Riemann–Stieltjes integrals with respect to it if one assumes that the integrand is smooth enough. If $H > \frac{1}{2}$ then the fractional Brownian motion has zero quadratic variation and there are various ways to define the integral (cf. Dai and Heyde [12], Föllmer [22] and Lin [32]). To our knowledge there are only two approaches that are applicable for the whole range $H \in (0,1)$, viz. the Hölder approach of Zähle [57] based on the fractional integration by parts formula (4.2) and the p-variation approach introduced by Young [56] and developed in [18, 19, 35].

Young [56] noted that the Riemann–Stieltjes integral can be extended to functions that are "together" smooth enough in the p-variation sense.

Theorem 6.1. *Let f and g be real functions over the interval $[0,1]$. Suppose that $f \in \mathscr{W}_p$ and $g \in \mathscr{W}_q$ for some p and q satisfying $\frac{1}{p} + \frac{1}{q} > 1$. Assume further that g is continuous. Then the Riemann–Stieltjes integral*

$$\int_0^1 f(t)\,\mathrm{d}g(t)$$

exists.

Applied to the fractional Brownian motion Theorem 6.1 yields:

Theorem 6.2. *Let Z be a fractional Brownian motion with Hurst index H. Let u be a stochastic process with sample paths almost surely in $\mathscr{W}_q$ with some q satisfying $q < 1/(1-H)$. Then the integral*

$$\int_0^1 u_t\,\mathrm{d}Z_t$$

exists almost surely in the Riemann–Stieltjes sense.

By Theorem 6.2 and Lemma 3.6 (some additional work is required regarding the Hölder continuity, cf. Zähle [57]) we have the following.

Corollary 6.3. *Let Z be a fractional Brownian motion with Hurst index H. Suppose that u is a stochastic process that has almost surely λ-Hölder continuous sample paths with some $\lambda > 1-H$. Then the integral*

$$U_t \;:=\; \int_0^t u_s\,\mathrm{d}Z_s \tag{6.1}$$

exists almost surely as a limit of Riemann–Stieltjes sums. Furthermore, the process U is almost surely β-Hölder continuous with any $\beta < H$.

Since the pathwise integral is a Riemann–Stieltjes integral it obeys the classical change of variables formula.

Theorem 6.4. *Let Z be a fractional Brownian motion with Hurst index H. Let $F \in C^{1,1}([0,1]\times\mathbb{R})$ such that for all $t \in [0,1]$ the mapping $t \mapsto \frac{\partial F}{\partial x}(t, Z_t)$ is in $\mathscr{W}_q$ for some $q < 1/(1-H)$. Then for all $s,t \in [0,1]$*

$$F(t,Z_t) - F(s,Z_s) \;=\; \int_s^t \frac{\partial F}{\partial x}(u,Z_u)\,\mathrm{d}Z_u + \int_s^t \frac{\partial F}{\partial t}(u,Z_u)\,\mathrm{d}u \tag{6.2}$$

almost surely.

Let us note the class of integrands is very restrictive if $H \le \frac{1}{2}$. Indeed, suppose that u is λ-Hölder continuous with some $\beta > 1-H$. Then

$$U_t \;:=\; \int_0^t u_s \,\mathrm{d}Z_s$$

exists by Corollary 6.3. However, the iterated integral

$$\int_0^t U_s \,\mathrm{d}Z_s \;=\; \int_0^t \int_0^s u_v \,\mathrm{d}Z_v \,\mathrm{d}Z_s$$

may not exist. Indeed, consider the integral

$$\int_0^1 Z_t \,\mathrm{d}Z_t \;=\; \int_0^1 \int_0^t \mathrm{d}Z_s \,\mathrm{d}Z_t.$$

If $H > \frac{1}{2}$ then Corollary 6.3 is applicable and (6.2) yields

$$\int_0^1 Z_t \,\mathrm{d}Z_t \;=\; \frac{1}{2} Z_1^2. \tag{6.3}$$

On the other hand, for $H \le \frac{1}{2}$ the pathwise integral does not exist. To see this take a partition $\pi = \{t_k : 0 = t_0 < \cdots < t_n = 1\}$ and note that

$$|\Delta Z_{t_k}|^2 \;=\; Z_{t_k}\Delta Z_{t_k} - Z_{t_{k-1}}\Delta Z_{t_k}.$$

Denote by Z_{t+} and Z_{t-} the right and left sided limits of Z at point t, resprectively. Then summing over $t_k \in \pi$ and letting $|\pi|$ tend to zero we see that if the Riemann–Stieltjes integral exists we must have

$$v_2^0(Z) \;=\; \int_0^1 Z_{t+} \,\mathrm{d}Z_t - \int_0^1 Z_{t-} \,\mathrm{d}Z_t \;=\; 0.$$

However, $v_2^0(Z)$ does not exist for $H < \frac{1}{2}$ and in the Brownian case we have $v_2^0(Z) = 1$.

Finally let us note that unlike the Ito integral the pathwise integral is not centred. Indeed, for $H > \frac{1}{2}$ the equation (6.3) implies

$$\mathbf{E} \int_0^1 Z_t \,\mathrm{d}Z_t \;=\; \frac{1}{2}.$$

7. Divergence integrals

The Wiener integral can be extended to non-deterministic integrands by using the stochastic calculus of variations, or Malliavin calculus. The extended integral turns out to be very different from the pathwise integral. This Malliavin calculus approach was proposed by Decreusefond and Üstünel [15] and was further studied e.g. by Alòs *et al.* [1, 2, 3]. Let us also note that essentially the same notion of integral may be obtained by using fractional white noise analysis (cf. Duncan *et al.* [20] and Bender [5]).

We recall some preliminaries of Malliavin calculus. For details on the topic we refer to Nualart [42].

Let $X = (X_t)_{t\in[0,1]}$ be a centred Gaussian process with covariance function R. Suppose that X is defined in a complete probability space $(\Omega, \mathscr{F}, \mathbf{P})$ where the σ-algebra $\mathscr{F}$ is generated by X. Let the Hilbert space $\mathscr{H}$ be $\mathscr{E}$ completed with respect to the inner product $\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathscr{H}} := R(t,s)$. So the mapping $\mathbf{1}_{[0,t]} \mapsto X_t$ extends to an isometry between $\mathscr{H}$ and $\mathscr{H}_1$. We denote by $X(\phi)$ the image of $\phi \in \mathscr{H}$ in this isometry.

Definition 7.1. Let $\mathscr{S}$ be the set of random variables of the form

$$F \;=\; f\left(X(\phi_1), \ldots, X(\phi_n)\right)$$

where $f \in C_b^\infty(\mathbb{R}^n)$, i.e. f and all its derivatives are bounded. The *Malliavin derivative* of $F \in \mathscr{S}$ is the $\mathscr{H}$-valued random variable

$$DF \;:=\; \sum_{k=1}^n \frac{\partial f}{\partial x_k}\left(X(\phi_1), \ldots, X(\phi_n)\right)\phi_k.$$

The derivative operator D is a closable unbounded operator from $L^p(\Omega)$ into $L^p(\Omega;\mathscr{H})$. In a similar way, the k times iterated derivative operator D^k maps $L^p(\Omega)$ into $L^p(\Omega;\mathscr{H}^{\otimes k})$.

Definition 7.2. The *domain* of D^k in $L^p(\Omega)$, denoted by $\mathbb{D}^{k,p}$, is the closure of $\mathscr{S}$ with respect to the norm

$$\|F\|_{k,p}^p \;:=\; \mathbf{E}\left(|F|^p + \sum_{j=1}^k \|D^jF\|^p_{\mathscr{H}^{\otimes j}}\right).$$

The divergence operator δ is introduced as the adjoint of D.

Definition 7.3. The *domain* of δ, $\operatorname{Dom}\delta$, is the set of $\mathscr{H}$-valued random variables u satisfying, for all $F \in \mathscr{S}$,

$$\left|\mathbf{E}\left\langle DF, u\right\rangle_{\mathscr{H}}\right|^2 \;\le\; c\,\mathbf{E}\,F^2$$

where c is a constant depending on u. If $u \in \operatorname{Dom}\delta$ then the *divergence* $\delta(u)$ is a square integrable random variable defined by the duality relation

$$\mathbf{E}\,\delta(u)F \;=\; \mathbf{E}\left\langle DF, u\right\rangle_{\mathscr{H}}$$

for all test variables $F \in \mathbb{D}^{1,2}$.

Given a separable Hilbert space V one can define the spaces $\mathbb{D}^{k,p}(V)$ of V-valued random variables as in Definition 7.2. A basic result of Malliavin calculus states that $\mathbb{D}^{1,2}(\mathscr{H}) \subset \operatorname{Dom}\delta$.

In the case of Brownian motion we have $\mathscr{H} = L^2([0,1])$. So DF is a square integrable stochastic process $(D_tF)_{t\in[0,1]}$. For $\phi \in L^2([0,1])$

$$\delta(\phi) \;=\; W(\phi) \;=\; \int_0^1 \phi(t)\,\mathrm{d}W_t.$$

In general, the divergence is an extension of the Ito integral and coincides with the stochastic integral introduced by Skorokhod [53]. Therefore, the divergence is also called the *Skorohod integral*.

In what follows, D and δ refer to the derivative and divergence, respectively, connected to the Brownian motion. The corresponding operators with respect to fractional Brownian motion Z are denoted by D^Z and δ^Z, respectively. We shall also use the notation

$$\int_0^t u_s\,\delta W_s \;:=\; \delta\big(u\mathbf{1}_{[0,t]}\big).$$

By Section 5 we know the form of the space $\mathscr{H}$ for the fractional Brownian motion: $\mathscr{H}$ is the completion of Λ and

$$\langle f,g\rangle_{\mathscr{H}} \;=\; \langle \mathrm{K}f,\mathrm{K}g\rangle_{L^2([0,1])}.$$

If $H \le \frac12$ then $\mathscr{H} = \Lambda$. For $H > \frac12$ the inclusion $\Lambda \subset \mathscr{H}$ is strict.

We can represent δ^Z in terms of δ (cf. [2]).

Proposition 7.4. *Let u be such a process that $\mathrm{K}u \in \operatorname{Dom}\delta$. Let Z be a fractional Brownian motion. Then*

$$\int_0^t u_s\,\delta Z_s \;:=\; \delta^Z\big(u\mathbf{1}_{[0,t]}\big) \;=\; \int_0^t (\mathrm{K}u)_s\,\delta W_s. \tag{7.1}$$

Actually, we may take (7.1) as the definition. The reason why we cannot use the Ito integral is that for non-deterministic u the process $\mathrm{K}u$ in (7.1) is measurable with respect to $\mathscr{F}_t$ only.

Let us note that the Skorohod integral is centred, i.e. $\mathbf{E}\,\delta^Z(u) = 0$.

The Ito formula for Skorokhod integrals takes the following form. For the proof of it we refer to Bender [6].

Theorem 7.5. *Suppose that $F \in C^{1,2}([0,1]\times\mathbb{R})$ satisfies*

$$\max\left(\Big|F(t,x)\Big|,\Big|\frac{\partial F}{\partial t}(t,x)\Big|,\Big|\frac{\partial F}{\partial x}(t,x)\Big|,\Big|\frac{\partial^2 F}{\partial x^2}(t,x)\Big|\right) \;\le\; Ce^{\lambda x^2}$$

where C *and* λ *are positive constants and* $\lambda < \frac{1}{4}$*. Let* $s, t \in [0,1]$*. Then the equation*

$$\begin{aligned} F(t, Z_t) - F(s, Z_s) &= \int_s^t \frac{\partial F}{\partial x}(u, Z_u)\, \delta Z_u + \int_s^t \frac{\partial F}{\partial t}(u, Z_u)\, \mathrm{d}u \\ &\quad + H \int_s^t \frac{\partial^2 F}{\partial x^2}(u, Z_u) u^{2H-1}\, \mathrm{d}u \end{aligned} \tag{7.2}$$

holds in $L^2(\Omega)$*.*

Using (7.2) we see that for every $H \in (0,1)$

$$\int_0^t Z_s\, \delta Z_s = \frac{1}{2} Z_t^2 - \frac{1}{2} t^{2H}$$

(recall that the corresponding pathwise integral does not exist for $H \le \frac{1}{2}$).

The Skorohod integral can be considered as a limit of Riemann–Stieltjes sums if one replaced the ordinary product by the so-called Wick product.

Definition 7.6. Let ξ be a centred Gaussian random variable. Its *Wick exponential* is

$$e^{\diamond \xi} := e^{\xi - \frac{1}{2} \mathbf{E}\, \xi^2}.$$

A random variable $X \diamond Y$ is the *Wick product* of X and Y if

$$\mathbf{E}\,(X \diamond Y) e^{\diamond W(\phi)} = \mathbf{E}\, X e^{\diamond W(\phi)} \cdot \mathbf{E}\, Y e^{\diamond W(\phi)}$$

for all $\phi \in L^2([0,1])$.

For the proof of the following we refer to Alós *et al.* [2], Proposition 4.

Proposition 7.7. *Let* u *be a stochastic process that is* β*-Hölder continuous in the norm of* $\mathbb{D}^{1,2}$ *with some* $\beta > |H - \frac{1}{2}|$*. Then*

$$\int_0^1 u_t\, \delta Z_t = \lim_{|\pi| \to 0} \sum_{t_k \in \pi} u_{t_{k-1}} \diamond \Delta Z_{t_k}$$

where the convergence is in $L^2(\Omega)$*.*

Let us consider the connection between the Skorohod and pathwise integral. We assume that $H > \frac{1}{2}$. Let $|\mathscr{H}| \subset \mathscr{H}$, $|\mathscr{H}|^{\otimes 2} \subset \mathscr{H}^{\otimes 2}$ consist of functions satisfying

$$\begin{aligned}
\|f\|^2_{|\mathscr{H}|} &:= H(2H-1)\int_0^1\int_0^1 |f(t)f(s)||t-s|^{2H-2}\,\mathrm{d}s\,\mathrm{d}t \\
&< \infty, \\
\|f\|^2_{|\mathscr{H}|^{\otimes 2}} &:= H^2(2H-1)^2 \times \\
&\quad \int_0^1\int_0^1\int_0^1\int_0^1 |f(r,s)f(t,u)|\left(|r-t||s-u|\right)^{2H-2}\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u \\
&< \infty,
\end{aligned}$$

respectively. Let $\mathbb{D}^{1,2}(|\mathscr{H}|)$ be the space of processes u satisfying

$$\mathbf{E}\Big(\|u\|^2_{|\mathscr{H}|} + \|D^Z u\|^2_{|\mathscr{H}|^{\otimes 2}}\Big) \ < \ \infty.$$

Proposition 7.8. *Let u be a stochastic process in $\mathbb{D}^{1,2}(|\mathscr{H}|)$ satisfying*

$$\int_0^1\int_0^1 \left|D^Z_s u_t\right| |t-s|^{2H-2}\,\mathrm{d}s\,\mathrm{d}t \ < \ \infty.$$

Then the pathwise Riemann–Stieltjes integral exists and

$$\int_0^1 u_t\,\mathrm{d}Z_t \ = \ \int_0^1 u_1\,\delta Z_t + H(2H-1)\int_0^1\int_0^1 D^Z_s u_t\,|t-s|^{2H-2}\,\mathrm{d}s\,\mathrm{d}t.$$

For the proof of Proposition 7.8 we refer to Alós *et al.* [3], Proposition 3 and Remark 1.

8. Summaries of included articles

[a] Fractional Brownian motion, random walks and binary market models. In the classical *Black–Scholes* pricing model two assets are traded continuously over the time interval $[0,T]$. Denote by B the riskless asset, or *bond,* and by S the risky asset, or *stock.* The dynamics of the assets are given by

$$\mathrm{d}B_t \ = \ B_t r(t)\,\mathrm{d}t, \tag{8.1}$$

$$\mathrm{d}S_t \ = \ S_t a(t)\,\mathrm{d}t + S_t\sigma\,\mathrm{d}W_t. \tag{8.2}$$

Here r is a deterministic interest rate, $\sigma > 0$ is a constant and W is a Brownian motion. The function a is the deterministic drift of the stock.

In some empirical studies on financial time series it has been demonstrated that the log-returns have a long-range dependence property (cf. Mandelbrot

[33] or Shiryaev [52]). The easiest way to implement this dependence to the Black–Scholes model is to replace the Brownian motion in (8.2) by a fractional one with Hurst index $H > \frac{1}{2}$:

$$\mathrm{d}S_t \;=\; S_t a(t)\,\mathrm{d}t + S_t\sigma\,\mathrm{d}Z_t. \tag{8.3}$$

The stochastic differential equation (8.3) is understood in the pathwise sense (which is possible if $H > \frac{1}{2}$). The pricing model with dynamics (8.1) and (8.3) is called the *fractional Black–Scholes* model.

Since the fractional Brownian motion is not a semimartingale one would expect that the fractional Black–Scholes model admits arbitrage opportunities. Indeed, such opportunities have been constructed e.g. by Cheridito [9], Dasgupta and Kallianpur [14], Rogers [46], Salopek [47] and Shiryaev [50]. To better understand what gives rise to the arbitrage we consider a discrete time (and whence a semimartingale) approximation of the fractional Black–Scholes model à la Cox–Ross-Rubinstein.

In the classical *Cox–Ross–Rubinstein* pricing model the assets are traded on time points $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = T$. The dynamics are

$$B_k^{(n)} \;=\; \left(1 + r_k^{(n)}\right) B_{k-1}^{(n)}, \tag{8.4}$$

$$S_k^{(n)} \;=\; \left(a_k^{(n)} + \left(1 + \sigma\xi_k^{(n)}\right)\right) S_{k-1}^{(n)}. \tag{8.5}$$

Here $B_k^{(n)}$ and $S_k^{(n)}$ are the values of the bond and stock, respectively, on $[t_k^{(n)}, t_{k+1}^{(n)})$. Similarly $r_k^{(n)}$ and $a_k^{(n)}$ are the interest rate and the drift of the stock, respectively, in the corresponding interval. The process $\xi^{(n)} = (\xi_k^{(n)})_{k\in\mathbb{N}}$ is a sequence of i.i.d. random variables with

$$\mathbf{P}(\xi_k^{(n)} = -1) \;=\; \frac{1}{2} \;=\; \mathbf{P}(\xi_k^{(n)} = 1).$$

By Donsker's theorem the random walk

$$W_t^{(n)} := \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i^{(n)}.$$

converges weakly to W (see e.g. [8]). So setting

$$r_k^{(n)} \;:=\; \frac{1}{n} r(Tk/n),$$

$$a_k^{(n)} \;:=\; \frac{1}{n} a(Tk/n),$$

it follows that $(B^{(n)}, S^{(n)})$ converges weakly to (B, S) as n increases.

To construct a fractional analogue of the Cox–Ross–Rubinstein model we need a fractional Donsker theorem. Set

$$Z_t^{(n)} \;:=\; \int_0^t z^{(n)}(t,s)\,\mathrm{d}W_s^{(n)} \tag{8.6}$$

where the function $z^{(n)}(t,\cdot)$ is an approximation to $z(t,\cdot)$, viz.

$$z^{(n)}(t,s) \;:=\; n\int_{s-\frac{1}{n}}^{s} z(\tfrac{\lfloor nt\rfloor}{n},u)\,\mathrm{d}u.$$

The reason why we use the approximation $z^{(n)}$ instead of z in (8.6) is that we want $Z^{(n)}$ to be a piecewise constant process, or a "disturbed" random walk.

Theorem 1 of [a] states that for $H > \frac{1}{2}$ the "disturbed" random walk $Z^{(n)}$ converges weakly to the fractional Brownian motion. Let us also note that, in different forms, the weak convergence to fractional Brownian motion have been studied before by e.g. Beran [7], Cutland *et al.* [11], Dasgupta [13] and Taqqu [54]. Our approximation scheme was motivated by the financial applications.

In the *fractional Cox–Ross–Rubinstein* model, or fractional binary model, the bond $B^{(n)}$ is given by (8.4). The stock price $S^{(n)}$ is given by (8.5) where

$$\xi_k^{(n)} \;:=\; \sigma\Delta Z_{Tk/n}^{(n)}.$$

By Theorem 4 of [a] the price processes $B^{(n)}$ and $S^{(n)}$ converge weakly to the corresponding price processes B and S of the fractional Black–Scholes model.

By Theorem 5 of [a] the model $(B^{(n)},S^{(n)})$ admits arbitrage opportunities if $n \geq n_H$. The arbitrage is due to the fact that if the stock price has had an upward (resp. downward) run long enough it will keep on going up (resp. down) for some time.

[b] Path space large deviations of a large buffer with Gaussian input traffic. A teletraffic model based on fractional Brownian motion and the corresponding storage system as introduced by Norros [38]. In this model the storage is fed by the fractional Brownian motion Z and the output rate is taken to be the unit. So, the *normalised fractional Brownian storage* V is given by the Reich formula

$$V_t \;=\; \sup_{s\leq t}\left(Z_t - Z_s - (t-s)\right).$$

The storage process V is non-negative and stationary.

Norros [39] studied the asymptotics of the busy periods of V. In [b] we generalise the results of [39] to a setting where the input rate is not quite the fractional Brownian motion, but behaves as one in the large time scales. More specifically, the input rate Z is a centred Gaussian process with stationary increments and variance

$$\mathbf{Var}\, Z_t \;=\; L(t)|t|^{2H}.$$

Here $H \in (0,1)$ and L is an even function satisfying

$$\lim_{\alpha\to\pm\infty} \frac{L(\alpha t)}{L(\alpha)} \;=\; 1$$

for all positive t, i.e. L is slowly varying at infinity. Note that if $L \equiv 1$ then the process Z is the fractional Brownian motion. If

$$L(t) \;=\; \sum_{k=1}^{n} a_k^2 |t|^{2(H_k - H_1)}$$

where $1 > H_1 > \cdots > H_n > 0$ then Z may be considered to be a superposition of n independent input streams with different Hurst indices, i.e.

$$Z \;=\; \sum_{k=1}^{n} a_k Z^{H_k}$$

where Z^{H_k} is a fractional Brownian motion with Hurst index H_k.

We are interested in the *busy periods* of V, i.e. its positive excursions around 0 (the present time) and in V_0, i.e. the *queue length*. We study them in the so-called large deviations framework. So we recall some preliminaries of this machinery, for details we refer to Dembo and Zeitouni [16].

Let Ω be a separable and complete metric space. A scaled family

$$\big(v(\alpha), X^{(\alpha)}\big)_{\alpha>0}$$

of Ω-valued random variables $X^{(\alpha)}$, $\alpha > 0$, satisfies the *large deviations principle* with *rate function* I if for all closed $F \subset \Omega$ and open $G \subset \Omega$ we have

$$\limsup_{\alpha\to\infty} \frac{1}{v(\alpha)} \ln \mathbf{P}\left(X^{(\alpha)} \in F\right) \;\le\; -\inf_{\omega\in F} I(\omega) \qquad \text{and}$$

$$\liminf_{\alpha\to\infty} \frac{1}{v(\alpha)} \ln \mathbf{P}\left(X^{(\alpha)} \in G\right) \;\ge\; -\inf_{\omega\in G} I(\omega).$$

In our case it is natural to take Ω to be the canonical space of paths $\omega : \mathbb{R} \to \mathbb{R}$ satisfying

$$\omega(0) = 0 \qquad \text{and} \qquad \lim_{t\to\pm\infty} \frac{|\omega(t)|}{1+|t|} = 0$$

equipped with the norm

$$\|\omega\|_\Omega \;:=\; \sup_{t\in\mathbb{R}} \frac{\omega(t)}{1+|t|}.$$

Let Z be the coordinate process, i.e. $Z_t(\omega) := \omega(t)$. Then the generalised Schilder theorem (cf. Deuschel and Stroock [17], Theorem 3.4.12) states that the scaled fractional Brownian motion $Z^{(\alpha)} := \alpha^{-1/2}Z$ satisfies the large deviations principle in Ω with scale $v(\alpha) = \alpha$ and rate function

$$I(\omega) \;=\; \begin{cases} \frac{1}{2}\|\omega\|^2_{\mathscr{R}} & \text{if } \omega \in \mathscr{R}, \\ \infty & \text{otherwise.} \end{cases}$$

Here $\mathscr{R}$ is the reproducing kernel Hilbert space of the fractional Brownian motion.

Let us denote

$$Z_t^{(\alpha)} \;:=\; \frac{1}{\alpha^H\sqrt{L(\alpha)}} Z_{\alpha t}$$

and

$$v(\alpha) \;:=\; \frac{\alpha^{2-2H}}{L(\alpha)}.$$

In article [b] we show that under some “metric entropy” conditions on L the processes $Z^{(\alpha)}$ converge weakly in Ω to the fractional Brownian motion and that the scaled family

$$\left(\tfrac{1}{\sqrt{v(\alpha)}} Z^{(\alpha)}, v(\alpha)\right)_{\alpha>0}$$

satisfies the large deviations principle in Ω with the same rate function as the fractional Brownian motion. As a consequence we obtain a result similar to that of Norros [39], viz.

$$\lim_{T\to\infty} \frac{L(T)}{T^{2-2H}} \ln \mathbf{P}(Z \in K_T) \;=\; -\inf_{\omega\in K_1} I(\omega).$$

Here K_T is the busy period of V at zero that is strictly longer than T. Also for the queue length $Q_x := \{V_0 > x\}$ we have

$$\lim_{x\to\infty} \frac{L(x)}{x^{2-2H}} \ln \mathbf{P}(Z \in Q_x) \;=\; -\inf_{\omega\in Q_1} I(\omega).$$

The constants $\inf\{I(\omega) : \omega \in K_1\}$ and $\inf\{I(\omega) : \omega \in Q_1\}$ are the same as in the fractional Brownian case, but the scale v depends on L.

[c] On Gaussian processes equivalent in law to fractional Brownian motion. The classical Hitsuda representation theorem (cf. Hida and Hitsuda [23]) states that a Gaussian process $\tilde{W}$ is equivalent in law to a standard Brownian motion if and only if it can be represented as

$$\tilde{W}_t \;=\; W_t - \int_0^t \int_0^s k(s,u)\,\mathrm{d}W_u\,\mathrm{d}s - \int_0^t a(s)\,\mathrm{d}s \tag{8.7}$$

for some Volterra kernel $k \in L^2([0,1]^2)$ and some function $a \in L^2([0,1])$. The Brownian motion W in (8.7) is constructed from $\tilde{W}$ by

$$W_t \;=\; \tilde{W}_t - \int_0^t \int_0^s r(s,u)(\mathrm{d}\tilde{W}_u - a(u)\,\mathrm{d}u)\,\mathrm{d}s - \int_0^t a(s)\,\mathrm{d}s. \tag{8.8}$$

Here r is the *resolvent* kernel of k.

In [c] we study to which extend representations similar to (8.7) and (8.8) hold for fractional Brownian motion. In particular, we study processes of the form

$$\tilde{Z}_t \;=\; Z_t - \int_0^t f(t,s)\,\mathrm{d}Z_s - A(t) \tag{8.9}$$

where Z is a fractional Brownian motion.

The slightly different form on (8.9) to (8.7) is due to the form on the reproducing kernel Hilbert space of the fractional Brownian motion (cf. Proposition 5.8). The difference is that (in addition to the Wiener integral) in (8.7) there is an ordinary integral and in (8.9) one has a $H+\frac{1}{2}$-fractional integral.

The space of Wiener integrands with respect to the fractional Brownian motion with Hurst index $H > \frac{1}{2}$ is incomplete. It follows that in this case there are Gaussian processes that are equivalent in law to the fractional Brownian motion but do not admit a representation (8.9). In the case $H \le \frac{1}{2}$ all Gaussian processes equivalent in law to the fractional Brownian motion can be represented in the form (8.9). In any case, Theorem 1 of [c] gives necessary and sufficient conditions on the kernel f and the function A in (8.9) under which the process $\tilde{Z}$ is equivalent in law to the fractional Brownian motion. The conditions are in terms of the operators K and K^* (cf. Section 4). We also show how one constructs the fractional Brownian motion Z in (8.9) from the process $\tilde{Z}$ by using Wiener integration and resolvent kernels.

We also consider briefly the Radon–Nikodym derivative of $\tilde{Z}$ with respect to Z.

As an application of the fractional Hitsuda representation (8.9) we consider a special kind of Gaussian equation

$$\zeta_t \;=\; \int_0^t z(t,s)\beta(s,\zeta)\,\mathrm{d}s + Z_t. \tag{8.10}$$

Here z is the kernel that transforms the standard Brownian motion into a fractional one, Z is a fractional Brownian motion and β is a non-anticipative functional. Theorem 2 of [c] states that under some integrability conditions of β the equation (8.10) has a Gaussian weak solution if and only if β is of the form

$$\beta(t,\omega) \;=\; \int_0^t k(t,u)\,\mathrm{d}\omega^*(u).$$

Here $k \in L^2([0,1]^2)$ and

$$\omega^*(t) \;=\; \int_0^t z^*(t,s)\,\mathrm{d}\omega(s)$$

where z^* is the kernel transforming the fractional Brownian motion into a standard one. If the Hurst index of the fractional Brownian motion in (8.10) satisfies $H \le \frac{1}{2}$ then the functional β can be represented as a “Wiener” integral

$$\beta(t,\omega) \;=\; \int_0^t f(t,s)\,\mathrm{d}\omega(s).$$

The kernel f is obtained from the kernel k as

$$f(t,s) \;=\; \mathrm{K}^* k(t,\cdot)\,(s).$$

In any case, if a Gaussian weak solution exists then so does a strong one and the latter is unique.

[d] On arbitrage and replication in the fractional Black–Scholes pricing model. Article [d] continues the theme of fractional Black–Scholes model and arbitrage studied in [a].

It has been proposed that the arbitrage in the fractional Black–Scholes model is due to the use of Riemann–Stieltjes integrals and would vanish if one uses Skorohod integrals and Wick products instead (cf. Hu and Øksendal [24]). Unfortunately, Skorohod integrals do not allow economical interpretation. In [d] we study the connection between the Skorohod self-financing and Riemann–Stieltjes self-financing conditions. In particular, we give an economical interpretation of the proposed arbitrage-free model in terms of Riemann–Stieltjes integrals.

Let the (discounted) stock price be given either by the Riemann–Stieltjes equation

$$\mathrm{d}S_t \;=\; S_t a^{\mathsf{RS}}(t)\,\mathrm{d}t + S_t\sigma\,\mathrm{d}\tilde{Z}_t \tag{8.11}$$

or by the (Wick–Ito–)Skorohod equation

$$\delta S_t \;=\; S_t a^{\mathsf{WIS}}(t)\,\mathrm{d}t + S_t\sigma\,\delta\tilde{Z}_t. \tag{8.12}$$

Here $\tilde{Z}$ is a fractional Brownian motion under the real world probability measure $\mathbf{P}$. Using the Riemann–Stieltjes Ito formula (6.2) and the Skorohod

Ito formula (7.2) to (8.11) and (8.12), respectively, we obtain the solutions

$$S_t = S_0 \exp\left(\int_0^t a^{\mathsf{RS}}(s)\,\mathrm{d}s + \sigma\tilde{Z}_t\right),$$

$$S_t = S_0 \exp\left(\int_0^t \left(a^{\mathsf{WIS}}(s) - \sigma^2 H s^{2H-1}\right)\,\mathrm{d}s + \sigma\tilde{Z}_t\right).$$

Consequently, equations (8.11) and (8.12) define the same model if (and only if)

$$a^{\mathsf{RS}}(t) = a^{\mathsf{WIS}}(t) - \sigma^2 H t^{2H-1}.$$

In the fractional Black–Scholes model there is no equivalent *martingale* measure. There is, however, a unique equivalent measure $\mathbf{Q}$ such that the solution to (8.11) or (8.12) is the *geometric fractional Brownian motion*

$$S_t := S_0 e^{Z_t - \frac{1}{2}t^{2H}}, \tag{8.13}$$

where Z is a $\mathbf{Q}$-fractional Brownian motion. Here we have taken $\sigma = 1$. For the Girsanov formula and the corresponding change of measure we refer to Norros et al. [41] in the Riemann–Stieltjes case and to Bender [6] in the Skorohod case.

Since the questions of arbitrage and replication are invariant under an equivalent change of measure we shall assume the model (8.13), i.e.

$$\mathrm{d}S_t = S_t\,\mathrm{d}\tilde{Z}_t - S_t H t^{2H-1}\,\mathrm{d}t,$$

$$\delta S_t = S_t\,\delta\tilde{Z}_t.$$

Let u be a *trading strategy,* i.e. u_t indicates the number of the shares of the stock owned by an investor at time t. Let v_t denote the bank account. The value $V_t(u)$ of the strategy at time t is of course

$$V_t(u) = u_t S_t + v_t$$

(it would be silly to use Wick products here). In the classical sense the strategy u is *self-financing* if the pathwise equation

$$\mathrm{d}V_t(u) = u_t\,\mathrm{d}S_t \tag{8.14}$$

holds. Using the Skorohod integral one can introduce a different self-financing condition, viz.

$$\delta V_t(u) = u_t\,\delta S_t. \tag{8.15}$$

When u satisfies (8.15) we call it *pseudo self-financing* and denote the corresponding wealth process by $V^{\mathsf{WIS}}(u)$. If u satisfies the classical condition (8.14) we denote the corresponding wealth by $V^{\mathsf{RS}}(u)$.

Using Proposition 7.8 we obtain Theorem 1 of [d] connecting the two self-financing conditions, viz.

$$V_T^{\mathsf{RS}}(u) - V_T^{\mathsf{WIS}}(u) \;=\; H(2H-1)\int_0^T S_t \int_0^T D_s^Z u_t \, |t-s|^{2H-2}\,\mathrm{d}s\,\mathrm{d}t.$$

For Markovian strategies $u_t = \gamma(t, S_t)$ Corollary 1 of [d] states that

$$V_T^{\mathsf{RS}}(\gamma) - V_T^{\mathsf{WIS}}(\gamma) \;=\; H\int_0^t \frac{\partial\gamma}{\partial x}(t,S_t)S_t^2 t^{2H-1}\,\mathrm{d}t. \tag{8.16}$$

With the pseudo self-financing condition one uses Skorohod integrals and can thus imitate the classical Ito calculus. In particular, with this way of calculating the fractional Black–Scholes model is free of arbitrage. This freedom of arbitrage follows basically from the fact that the Skorohod integrals are centred. As for replication consider a Markovian claim $f_T = f_T(S_T)$. Now

$$V_T^{\mathsf{WIS}}(\frac{\partial\gamma}{\partial x}) \;=\; \gamma(T,S_T) - \int_0^T \left(\frac{\partial\gamma}{\partial t}(t,S_t) + HS_t^2 t^{2H-1}\frac{\partial^2\gamma}{\partial x^2}(t,S_t)\right)\mathrm{d}t.$$

So if γ satisfies the *fractional Black–Scholes differential equation*

$$\frac{\partial\gamma}{\partial t}(t,x) \;=\; -Hx^2t^{2H-1}\frac{\partial^2\gamma}{\partial x^2}(t,x)$$

with the boundary condition $\gamma(T,x) = f_T(x)$ then $\frac{\partial\gamma}{\partial x}$ replicates the claim f_T. Moreover, $\gamma(0,S_0)$ is the corresponding fair price. Note however that by (8.16) this Skorohod replication is actually a super replication in the Riemann–Stieltjes sense if the claim f_T is convex.

9. Errata

Most (hopefully) of the typos and more serious mistakes of the included articles are collected here. In articles [c] and [d] the page numbers refer to the original preprints.

[a] Fractional Brownian motion, random walks and binary market models.

1. Page 346: The Riemann-sum argument to (4) in the proof of Theorem 1 of [a] is inaccurate since the kernel z is singular. For the accurate proof we refer to Nieminen [37].
2. Page 349: The last equation in the last equation array should be

$$= \;\mathbf{P}(\sup_{t\le T}|\Delta Z_t| \ge \frac{1}{2}).$$

3. Page 350: The line ending "...Since $\Delta Z_t^{(1,n)} < \frac{1}{2}$ the" should be "...Since $|\Delta Z_t^{(1,n)}| < \frac{1}{2}$ the"

[b] Path Space Large Deviations of a Large Buffer with Gaussian Input Traffic.

1. Page 121: In Remark 2.11 we assume that all the constant a_k of Example 2.10 are non-zero.
2. Page 127: The last two lines of the first equation array should be

$$\begin{aligned} &= \mathbf{P}\left(\sup_{t\le 0}\left(\frac{\sqrt{L(x)}}{x^{1-H}}Z_t^{(x)} - t\right) \ge 1\right) \\ &= \mathbf{P}\left(\frac{\sqrt{L(x)}}{x^{1-H}}Z^{(x)} \in Q_1\right). \end{aligned}$$

[c] On Gaussian prcoesses equivalent in law to fractional Brownian motion.

1. Page 1: Line ending "...Any Gaussian process $\tilde{Z} = (\tilde{Z}_t)_{t\in[0,1]}$" should be "...Any centred Gaussian process $\tilde{Z} = (\tilde{Z}_t)_{t\in[0,1]}$"
2. Page 2: The Brownian motion W in the first line is not the same as in Equation (1.1). The Brownian motion W in Equation (1.1) is constructed from $\tilde{W}$ by Equation (1.2). Similar confusion may be seen throughout the paper, i.e. the process Z is not given: $\tilde{Z}$ is given and a fractional Brownian motion Z is constructed from it.
3. Page 6: The definition of K^* should be

$$\mathrm{K}^* f(t) \;:=\; \frac{1}{c_3}\left(I_-^{-\alpha} s^\alpha f(s)\right)(t)$$

[d] On arbitrage and replication in the fractional Black–Scholes pricing model.

1. Page 2: In the p-variation integral it is also assumed that f and g have no common discontinuities.
2. Page 2: In last line "$p < 1/H$" should be "$p > 1/H$".
3. Page 4: Line ending "...Here c is a constant depending on F." should be "...Here c is a constant depending on u."
4. Page 5: Growth condition in Proposition 2 should be

$$\max\left(\Big|F(t,x)\Big|, \Big|\frac{\partial F}{\partial t}(t,x)\Big|, \Big|\frac{\partial F}{\partial x}(t,x)\Big|, \Big|\frac{\partial^2 F}{\partial x^2}(t,x)\Big|\right) \;\le\; Ce^{\lambda x^2}$$

5. Page 5: Before equation line

$$\mathrm{d}S_t \;=\; S_t\,\mathrm{d}W_t$$

there should be "taking $\sigma = 1$".

6. Page 6: Before equation line

$$S_t \;:=\; S_0 e^{Z_t - \frac{1}{2}t^{2H}}$$

there should be "taking $\sigma = 1$".
7. Page 7: The lines starting "for he wealth..." should be "for the wealth

$$V_t(u) \;=\; u_t S_t + v_t$$

of a trading portfotio u. Here v is the (discounted) bank account."
8. Page 7: The fifth equation should be

$$\mathbf{E}_{\mathbf{Q}} V_T^{\mathsf{WIS}}(u) \;=\; V_0^{\mathsf{WIS}}(u) \;=\; u_0 S_0 + v_0$$

9. Page 7: In equation after the equation (5.2) the variable of integration should be s (t is the upper limit).
10. Page 8: In Theorem 1 the formula should be

$$V_T^{\mathsf{RS}}(u) - V_T^{\mathsf{WIS}}(u) \;=\; H(2H-1)\int_0^T S_t \int_0^T D_s u_t \, |t-s|^{2H-2} \, \mathrm{d}s \, \mathrm{d}t.$$

11. Page 8: The initial value $V_0(u) = u_0 S_0 + v_0$ should be added to the right hand side of formulas (5.5) and (5.6).
12. Page 9: In Corollary 1 the formula should be

$$V_T^{\mathsf{RS}}(\gamma) - V_T^{\mathsf{WIS}}(\gamma) \;=\; H \int_0^T \frac{\partial \gamma}{\partial x}(t, S_t) S_t^2 t^{2H-1} \, \mathrm{d}t.$$

13. Page 10: The fourth line in the first equation array should be

$$= \; F(T, Z_T) - \int_0^T \left(\frac{\partial F}{\partial t}(t, Z_t) + H t^{2H-1} \frac{\partial^2 F}{\partial x^2}(t, Z_t) \right) \mathrm{d}t$$

14. Page 11: Line starting "Let us consider *Delta*-hedging..." should be "Let us consider Δ-hedging..."

References

a. Sottinen, T. (2001) *Fractional Brownian motion, random walks and binary market models.* Finance Stoch. **5**, no. 3, 343–355.
b. Kozachenko, Yu., Vasylyk, O. and Sottinen, T. (2002) *Path Space Large Deviations of a Large Buffer with Gaussian Input Traffic.* Queueing Systems **42**, no. 2, 113–129.
c. Sottinen, T. (2002) *On Gaussian processes equivalent in law to fractional Brownian motion.* University of Helsinki, Department of Mathematics, Preprint **328**, 17 p.

d. Sottinen, T. and Valkeila, E. (2002) *On arbitrage and replication in the Fractional Black-Scholes pricing model.* University of Helsinki, Department of Mathematics, Preprint **335**, 13 p.

1. Alòs, E., Mazet, O. and Nualart, D. (2000) *Stochastic Calculus with respect to fractional Brownian motion with Hurst index less than* $\frac{1}{2}$. Stoch. Proc. Appl. **86**, 121–139.
2. Alòs, E., Mazet, O. and Nualart, D. (2001) *Stochastic calculus with respect to Gaussian processes.* Ann. Probab. **29**, no. 2, 766–801.
3. Alòs, E. and Nualart, D. (2001) *Stochastic integration with respect to the fractional Brownian motion.* Preprint, 25 p.
4. Arcones, M. A. (1995) *On the law of the iterated logarithm for Gaussian processes.* J. Theor. Prob. **8**, no. 4, 877–904.
5. Bender, C. (2002) *An Ito Formula for Generalized Functionals of a Fractional Brownian Motion with Arbitrary Hurst Parameter.* Stochastic Process Appl., to appear.
6. Bender, C. (2002) *The fractional Ito integral, change of measure and absence of arbitrage.* Preprint, 34 p.
7. Beran, J. (1994) *Statistics for Long-Memory Processes.* Chapman & Hall, New York.
8. Billingsley, P. (1968) *Convergence of Probability Measures.* Chapman & Hall, New York.
9. Cheridito, P. (2001) *Regularizing Fractional Brownian Motion with a View towards Stock Price Modelling.* Ph.D. thesis, ETH Zürich.
10. Cornfeld, I. P., Fomin, S. V. and Sinai, Ya. G. (1982) *Ergodic Theory.* Springer.
11. Cutland, N.J., Kopp, P.E., Willinger, W. (1995) *Stock price returns and the Joseph effect: a fractional version of the Black–Scholes model.* Progress in Probability **36,** 327–351.
12. Dai, W. and Heyde, C. C. (1996) *Itô's Formula with respect to Fractional Brownian Motion and its Applications.* J. Appl. Stoch. Anal. **9**, 439–458.
13. Dasgupta, A. (1998) *Fractional Brownian motion: its properties and applications to stochastic integration.* Ph.D. thesis, University of North Carolina.
14. Dasgupta, A. and Kallianpur, G. (2000) *Arbitrage opportunities for a class of Gladyshev processes.* Appl. Math. Opt. **41**, 377–385.
15. Decreusefond, L. and Üstünel, A. S. (1999) *Stochastic analysis of the fractional Brownian motion.* Potential analysis **10**, 177–214.
16. Dembo, A. and Zeitouni, O. (1998) *Large Deviations Techniques and Applications. Second Edition.* Springer, New York.
17. Deuschel, J.-D., Stroock, D. (1984) *Large Deviations.* Academic Press, Boston MA.
18. Dudley, R. M. and Norvaiša, R. (1998) *An introduction to* p*-variation and Young integrals.* MaPhySto Lecture Notes **1**.
19. Dudley, R. M. and Norvaiša, R. (1999) *Differentiability of Six Operators on Nonsmooth Functions and* p*-Variation.* Springer, Berlin.

20. Duncan, T. E., Hu, Y. and Pasik-Duncan, B. (2000) *Stochastic calculus for fractional Brownian motion. I. Theory.* SICON **38**, 582–612.
21. Embrechts, P. and Maejima, M. (2002) *Selfsimilar Processes.* Princeton University Press, Princeton.
22. Föllmer, H. (1980) *Calcul d'Itô sans probabilités.* Séminaire de probabilités **XV**, 143–150, 1980.
23. Hida, T. and Hitsuda, M. (1993) *Gaussian Processes.* Translations of Mathematical Monographs, **120**, AMS, Providence, RI.
24. Hu, Y. and Øksendal, B. (1999) *Fractional white noise calculus and applications to finance.* Preprint, 33 p.
25. Hunt, G. A. (1951) *Random Fourier transforms.* Trans. Amer. Math. Soc. **71**, 38–69.
26. Hurst, H. E. (1951) *Long-term storage capacity of reservoirs.* Trans. Am. Soc. Civil Engineers **116**, 770–799.
27. Kallenberg, O. (1997) *Foundations of modern probability.* Springer, New York.
28. Kolmogorov, A. N. (1940) *Wienersche Spiralen und einige andere interessante Kurven in Hilbertschen Raum.* Comptes Rendus (Doklady) Acad. Sci. USSR (N.S.) **26**, 115–118.
29. Kolmogorov, A. N. (1941) *The local structure of turbulence in an incompressible fluid at very high Reynolds numbers.* Dokl. Acad. Nauk USSR **30**, 299–303.
30. Lamperti, J. (1962) *Semi-stable stochastic processes.* Trans. Amer. Math. Soc. **104**, 62–78.
31. Lévy, P. (1953) *Random functions: General theory with special reference to Laplacian random functions.* Univ. California Publ. Statist. **1**, 331–390.
32. Lin, S. J. (1995) *Stochastic analysis of fractional Brownian motions.* Stochastics and Stochastics Reports **55**, no. 1–2, 121–140.
33. Mandelbrot, B. (1997) *Fractals and Scaling in Finance. Discontinuity, Concentration, Risk.* Springer, New York.
34. Mandelbrot, B. and Van Ness, J. (1968) *Fractional Brownian motions, fractional noises and applications.* SIAM Review **10**, 422–437.
35. Mikosch, T. and Norvaiša, R. (2000) *Stochastic integral equations without probability.* Bernoulli **6**, no. 3, 401–434.
36. Molchan, G. and Golosov, J. (1969) *Gaussian stationary processes with asymptotic power spectrum.* Soviet Math. Dokl. **10**, 134–137.
37. Nieminen, A. (2002) *Fractional Brownian motion and martingale-differences.* Preprint, 10 p.
38. Norros, I. (1994) *A storage model with self-similar input.* Queueing Systems Theory Appl. **16**, no. 3-4, 387–396.
39. Norros I. (1999) *Busy Periods of Fractional Brownian Storage: A Large Deviations Approach.* Advances in Performance Analysis, **2**, 1–19.
40. Norros, I., Mannersalo, P. and Wang, J. (1999) *Simulation of Fractional Brownian Motion with Conditional Random Displacement.* Advances in Performance Analysis **2**, 77–101.

41. Norros, I., Valkeila E. and Virtamo J. (1999) *An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motion.* Bernoulli **5**, no. 4, 571–587.
42. Nualart, D. (1995) *The Malliavin Calculus and Related Topics.* Springer, New York.
43. Nuzman, C. J. and Poor, H. V. (2000) *Linear estimation of self-similar processes via Lamperti's transformation.* J. Appl. Prob. **37**, 429–452.
44. Pipiras, V. and Taqqu, M. (2000) *Integration questions related to fractional Brownian motion.* Probab. Theory Related Fields **118**, no. 2, 251–291.
45. Pipiras, V. and Taqqu, M. (2001) *Are classes of deterministic integrands for fractional Brownian motion on an interval complete?* Bernoulli **7**, no. 6 873–897.
46. Rogers, L. C. G. (1997) *Arbitrage with fractional Brownian motion.* Math. Finance **7**, 95–105.
47. Salopek, D. M. (1998) *Tolerance to arbitrage.* Stochastic Processes and their Applications **76**, 217–230.
48. Samko, S. G., Kilbas A. A. and Marichev, O. I. (1993) *Fractional integrals and derivatives. Theory and applications.* Gordon and Breach Science Publishers, Yverdon.
49. Samorodnitsky, G. and Taqqu, M. (1994) *Stable non-Gaussian random processes.* Chapman & Hall, New York.
50. Shiryaev, A. N. (1998) *On arbitrage and replication for fractal models.* MaPhySto Research Reports **20**.
51. Shiryaev, A. N. (1999) *Kolmogorov and the Turbulence.* MaPhySto Misc. **14**.
52. Shiryaev, A. N. (1999) *Essentials of stochastic finance. Facts, models, theory.* World Scientific, River Edge, NJ.
53. Skorokhod, A. V. (1975) *On a generalization of a stochastic integral.* Theory Probab. Appl. **20**, 219–233.
54. Taqqu, M. S. (1975) *Weak convergence to fractional Brownian motion and to the Rosenblatt process.* Z. Wahrsch. Verw. Geb. **31,** 287–302.
55. Yaglom, A. M. (1958) *Correlation theory of processes with random stationary nth increments.* AMS Transl. (2), 8, pp. 87–141.
56. Young, L. C. (1936) *An inequality of the Hölder type, connected with Stieltjes integration.* Acta Math. **67**, 251–282.
57. Zähle, M. (1998) *Integration with respect to fractal functions and stochastic calculus. I* Probab. Theory Related Fields **111**, no. 2, 333–347.